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Optimal Education and Pensions in an Endogenous Growth Model

Elena DEL REY
Miguel Angel LOPEZ-GARCIA

CORE
Université catholique de Louvain

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Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

1 Introduction

It is well known that in life-cycle growth models with only physical capital à la Diamond (1965), the laissez-faire equilibrium will generally not coincide with the optimum. The reason is that the capital-labour ratio associated with private saving will be different from the Golden Rule one. The implications for policy design are clear: whenever the capital-labour ratio is higher [resp. lower] than the optimum one, intergenerational transfers from [resp. to] younger to [resp. from] older generations are required. In the presence of endogenous human capital, however, a second potential departure from optimality arises, as individuals may not choose the correct amount of education investment. Thus, a second policy instrument will be required to achieve optimality, and a subsidy to human capital investments becomes a natural candidate.

The role that can be played by education subsidies and pensions has been the object of some interest. Boldrin and Montes (2005) show that, in the absence of credit markets, two systems of independent intergenerational transfers (to the young from the middle-aged and to the old from the same middle aged), can be used to replicate the laissez-faire equilibrium with credit markets (where the middle-aged lend to the young and receive the return of this investment when retired). They argue that, under reasonable values of the parameters, the laissez-faire equilibrium will exhibit a rate of return to capital higher than the growth rate of the economy, a situation that they associate with dynamic efficiency. One of the purposes of this paper is precisely to show that whether the rate of return to capital is higher or lower than the growth rate of output is no longer enough to assess the efficiency of the competitive equilibrium. The reason is that, even if physical capital investments are too low at the laissez-faire economy, human capital investments can be too large, and this leaves some scope for a Pareto improvement.

Clearly, in order to determine the optimal levels of physical and human capital and evaluate the scope for Pareto improvements, we need to adopt some welfare criterion. Caballé (1995) and Docquier, Paddison and Pestieau (2007), among others, posit that the objective of the social planner is to maximize a discounted sum of individual utilities defined over consumption levels per unit of *natural* labour. We adopt a different approach. In line with what Diamond (1965) and Buiter (1979) respectively do in the basic model without productivity growth and with exogenously given productivity growth, we embrace the Golden Rule criterion. This entails the search for the balanced growth path that maximizes the lifetime welfare of a representative individual subject to the constraint that everyone else's welfare is fixed at the same level. And we do so in the only sensible way in an endogenous growth framework, i.e., by considering a utility function whose arguments are individual consumptions per unit of *efficient* labour.

This social objective may seem awkward at first and it may be worth to devote a few comments to discuss its implications. With this purpose, let us abstract for a while from the overlapping generations structure of the model and assume for simplicity a constant labour force in an economy experiencing increases in labour productivity at an (exogenous or endogenous) rate. Along a balanced growth path, consumption per unit of natural labour will be increasing at this rate, so that consumption per unit of efficient labour will be kept constant. Under these circumstances, if a social planner had a utility function defined over consumption per unit of natural labour, it is clear that, for plausible specifications, the utility index would be growing without limit. Since utility will eventually be infinite along a balanced growth path, there is simply no scope for utility maximization. A way to sidestep this is to assume that the planner maximizes a discounted sum of utilities, and this is actually the standard procedure.

However, if the social planner's utility function is defined over consumption per unit of efficient labour, it is clear that the utility index will be finite along a balanced growth path. Hence, there is scope for optimizing this index, so that the Golden Rule, using as control variables consumptions per unit of labour efficiency, emerges as a reasonable alternative. Intuitively, our criterion is most pertinent if it is considered that, as productivity increases, individuals must "share in the growth of the economy", i.e., their consumption has to increase at the same rate in order to maintain their well-being. If we believe instead that satisfaction is determined by consumption levels alone, so that an increase in consumption raises welfare whatever the evolution of productivity, then the utility of consumption per unit of natural labour should the basis for an adequate social objective.[1] As it will be made clearer, both approaches are consistent with the same *ordinal* preferences of the individuals of a given generation. In contrast, the treatment of different generations will *not* be the same and, as a result, the adoption of one or the other normative approach will yield different allocations of resources at the optimum.

In this paper, we embrace this Golden Rule criterion. We identify the optimum, compare it to the laissez faire with perfect credit markets (where younger individuals can borrow to finance their education) and identify the optimal policy. Not surprisingly, one of the conditions of the optimal allocation is that the marginal product of physical capital (per unit of efficient labour) equals the (endogenous) growth rate of the economy. And, as far as human capital accumulation is concerned, a condition equating marginal benefits and marginal costs is obtained. It is shown that the laissez faire with perfect credit markets cannot possibly attain the Golden Rule. And finally, we prove that, in order to attain the Golden Rule, education subsidies should be negative, i.e., education should be *taxed* instead of subsidized. The reason is that individuals choose their human capital investments accounting only for the effects on their earnings and loan repayment costs. Thus, when facing the optimal (i.e., Golden Rule) wage and interest rates in the laissez faire, individuals will ignore the costs associated with maintaining these factor prices at their Golden Rule level when human capital increases. Under these circumstances, they over-invest in education, and a tax is required. However, out of the Golden Rule balanced growth path, positive subsidies to education can be welfare improving under certain conditions. Also, we show that, if education policy is modelled as subsidies to the repayment of the loans borrowed to finance education, pension benefits are to be strictly positive in order to attain the Golden Rule.

The rest of the paper is organized as follows. Section 2 sets up the general endogenous growth framework and its corresponding Golden Rule. Section 3 presents the decentralized market equilib-

[1] This point has been suggested to the authors by David de la Croix.

rium in the presence of government when the tax instruments available are lump-sum taxes on both the working and the retired population and education subsidies. Section 4 compares the laissez-faire balanced growth path with the Golden Rule, and derives some propositions on the efficiency or inefficiency of these equilibria. Section 5 characterizes the education subsidy and intergenerational transfers consistent with the Golden Rule balanced growth path. Section 6 analyzes the comparative dynamics in terms of both physical and human capital-labour ratios and welfare level associated with changes in the policy parameters along any arbitrary balanced growth path. It is shown, in particular, that the welfare effects of modifying the tax parameters depend, on the one hand, on the relationship between the interest rate and the growth rate of the economy and, on the other, on the impact of the individual investment in education on the present value of her lifetime resources. Section 7 concludes and more technical details are displayed in the Appendix.

2 The Model and the Planner's Optimum

The basic framework of analysis is the overlapping generations model with both human and physical capital developed in Boldrin and Montes (2005) and Docquier et al. (2007). At period t, L_{t+1} individuals are born. They coexist with L_t middle-aged and L_{t-1} old-aged. Population grows at the exogenous rate n so that $L_t = (1+n)L_{t-1}$ with $n > -1$. Agents are born with an endowment of basic "knowledge" $\overline{h}^y_{t-1}$, which is measured in units of efficient labour per unit of natural labour, and is an input in the production of future human capital h_t. We assume that $\overline{h}^y_{t-1}$ is an exogenously given fraction μ_{t-1} of the existing level of knowledge, i.e., $\overline{h}^y_{t-1} = \mu_{t-1}h_{t-1}$. Human capital is produced out of the amount of output invested in education d_{t-1} and basic knowledge $\overline{h}^y_{t-1}$ according to the production function $h_t = E(d_{t-1}, \overline{h}^y_{t-1})$. Assuming constant returns to scale, the production of human capital can be written in intensive terms as $h_t/\overline{h}^y_{t-1} = e(\tilde{d}_{t-1})$, where $e(.)$ satisfies the Inada conditions and $\tilde{d}_{t-1} = d_{t-1}/\overline{h}^y_{t-1}$ is the amount of output devoted to education per unit of inherited human capital.

A single good Y_t is produced by means of physical capital K_t and human capital H_t, according to a constant returns to scale production function $Y_t = F(K_t, H_t)$. As explained below, only the middle-aged work and they inelastically supply one unit of natural labour, so that $H_t = h_t L_t$. Physical capital is assumed to fully depreciate each period. If we define $k_t = K_t/L_t$ as the physical capital per unit of natural labour ratio and $\tilde{k}_t = K_t/H_t = k_t/h_t$ as the physical capital per unit of efficient labour ratio, the technology can be described as $Y_t/H_t = f(\tilde{k}_t)$, where $f(.)$ also satisfies the Inada conditions.

The lifetime utility function of an individual born at period $t-1$ is $U_t = U(c^m_t, c^o_{t+1})$, where c^m_t and c^o_{t+1} denote her consumption levels as middle-aged and old-aged, respectively. This utility function is assumed to be strictly quasi-concave and homogeneous of degree $j > 0$. The reason why we only impose strict quasi-concavity instead of strict concavity of the individual utility function is that, as it will be made clearer shortly, we are only interested in *ordinal* preferences. This is a less stringent assumption than the one made in Docquier et al. (2007), who need to assume that

$0 < j < 1$ in order for the discounted sum of individual utilities to be well defined.

Total output produced in period t, $F(K_t, H_t)$, can be devoted to consumption, $c_t^m L_t + c_t^o L_{t-1}$, investment in human capital, $d_t L_{t+1}$, and investment in physical capital, K_{t+1}. Thus, the aggregate feasibility constraint writes

$$F(K_t, H_t) = c_t^m L_t + c_t^o L_{t-1} + d_t L_{t+1} + K_{t+1} \quad (1)$$

or, expressed in units of natural labour:

$$h_t f(k_t/h_t) = c_t^m + \frac{c_t^o}{1+n} + (1+n)d_t + (1+n)k_{t+1} \quad (2)$$

Alternatively, we can divide (2) by h_t, which is given at time t, and obtain the aggregate feasibility constraint in period t measured in terms of output per unit of efficient labour:

$$f(\tilde{k}_t) = \tilde{c}_t^m + \frac{\tilde{c}_t^o}{\mu_{t-1} e(\tilde{d}_{t-1})(1+n)} + (1+n)\mu_t \tilde{d}_t + \mu_t e(\tilde{d}_t)(1+n)\tilde{k}_{t+1} \quad (3)$$

where $\tilde{c}_t^m = c_t^m/h_t$ and $\tilde{c}_t^o = c_t^o/h_{t-1}$ denote respectively consumption when middle-aged and consumption when old-aged per unit of efficient labour.[2] Note that $h_{t+1}/h_t = \mu_t e(\tilde{d}_t) = 1 + g_{t+1}$, where g_{t+1} is the growth rate of productivity from period t to period $t+1$.

Along a balanced growth path, all variables expressed in terms of output per unit of natural labour are growing at rate g. In consequence, all variables expressed in terms of output per unit of efficient labour remain constant: $\tilde{c}_t^m = \tilde{c}_{t+1}^m = \tilde{c}^m$, $\tilde{c}_t^o = \tilde{c}_{t+1}^o = \tilde{c}^o$, $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}$ and $\tilde{d}_t = \tilde{d}_{t+1} = \tilde{d}$.

We can now turn to the discussion of the planner's objective. In the presence of productivity growth that translates into consumption growth (as is, of course, the case along a balanced growth path), consumption levels will grow without limit. This means that we cannot choose the consumption levels that maximize $U_t = U(c_t^m, c_{t+1}^o)$, so that a different approach is required. Since the utility function is homogeneous, and to the extent that h_t is exogenously given at the beginning of each period, we can write:

$$\tilde{U}_t = U(\tilde{c}_t^m, \tilde{c}_{t+1}^o) = U\left(c_t^m/h_t, c_{t+1}^o/h_t\right) = (1/h_t^j)U(c_t^m, c_{t+1}^o) = (1/h_t^j)U_t \quad (4)$$

Thus, a "new" utility function is obtained by means of a monotonic transformation of the first one, thus ensuring that *ordinal* preferences are respected. Notice that this utility function has the same functional form as the original one and, consequently, continues to be homogeneous of degree j. Also, the slope, curvature and higher derivatives of indifference curves in $(\tilde{c}_t^m, \tilde{c}_{t+1}^o)$ space are the same as those of the corresponding indifference curves in (c_t^m, c_{t+1}^o) space. In particular, they will be independent of the level of labour efficiency.

Therefore, the arguments of the "new" utility function in (4) are consumptions per unit of efficient labour. We can now posit that the social planner's objective is to choose the balanced

[2]Note that $c_t^m L_t$ and $c_t^o L_{t-1}$ are expressed in units of output. Since middle-aged individuals supply one unit of natural labour, c_t^m and c_t^o are expressed in units of output per unit of *natural* labour. The interpretation of $\tilde{c}_t^m$ and $\tilde{c}_t^o$ in terms of units of output per unit of *efficient* labour follows naturally.

growth path that maximizes the welfare of a representative individual subject to the constraint that everyone attains the same utility level, i.e., $U(\tilde{c}^m, \tilde{c}^o)$. We adopt this approach, which is reminiscent of Diamond (1965)'s original treatment of the Golden Rule in an OLG framework with productive capital.[3] Then, the social planner will choose $(\tilde{c}^m, \tilde{c}^o, \tilde{k}, \tilde{d})$ that maximize $U(\tilde{c}^m, \tilde{c}^o)$ subject to the balanced growth path version of (3).[4]

From the first order conditions we obtain:

$$\frac{\partial U(\tilde{c}^m_*, \tilde{c}^o_*)/\partial \tilde{c}^m}{\partial U(\tilde{c}^m_*, \tilde{c}^o_*)/\partial \tilde{c}^o} = (1+g_*)(1+n) \tag{5}$$

$$f'(\tilde{k}_*) = (1+g_*)(1+n) \tag{6}$$

$$e'(\tilde{d}_*)\left(\frac{\tilde{c}^o_*}{[(1+g_*)(1+n)]^2} - \tilde{k}_*\right) = 1 \tag{7}$$

$$\tilde{c}^m_* + \frac{\tilde{c}^o_*}{(1+g_*)(1+n)} = f(\tilde{k}_*) - (1+g_*)(1+n)\tilde{k}_* - (1+n)\mu\tilde{d}_* \tag{8}$$

$$1 + g_* = \mu e(\tilde{d}_*) \tag{9}$$

Definition 1 *The* Golden Rule *balanced growth path* $(\tilde{c}^m_*, \tilde{c}^o_*, \tilde{k}_*, \tilde{d}_*)$ *provides the maximum level of welfare that can be achieved by a representative individual, subject to the feasibility constraint and the additional constraint that everyone else attains the same level. It is characterized by expressions (5)-(9).*

The interpretation of these equations is simpler if we start from the exogenous productivity growth setting, that we obtain for a given g and (without loss of generality) $\tilde{d} = 0$. Then, the Golden Rule is characterized by (5), (6) and (8) with g given and $\tilde{d} = 0$ (see Buiter, 1979). Equation (5) is the equality of the marginal rate of substitution between second and third period consumptions and the counterpart in the current model of the so-called "biological" interest rate, i.e., the economy's growth rate. Equation (6) is the equality of the marginal product of physical capital (per unit of efficient labour) and the growth rate of labour measured in efficiency units (i.e., the sum of the rates at which the efficiency of labour and the natural units of labour respectively grow). Of course, if $g = 0$ we are back to Diamond's framework and we obtain the original Golden Rule.

[3] Thus, in the pure tradition of the Golden Rule approach, we take sides with Samuelson in his old controversy with Lerner published in the Journal of Political Economy, Vol. 67 No. 5 (Oct. 1959) pp. 512-525. The alternative approach would emphasize instead the allocation of consumption across different generations concurring at time t.

[4] Alternatively, we could state this problem in terms of the choice of variables expressed in units of *natural* labour. At every period along a balanced growth path, since h_t is the consequence of a past decision, the planner chooses ($c^m_t, c^o_{t+1}, k_{t+1}$ and h_{t+1}) that maximize $U(c^m_t/h_t, c^o_{t+1}/h_t)$ subject to $c^m_t/h_t + (c^o_{t+1}/h_t)(h_t/h_{t+1})/(1+n) = f(k_{t+1}/h_{t+1}) - (1+n)\mu_t e^{-1}(h_{t+1}/\mu_t h_t) - (1+n)k_{t+1}/h_t$. Notice that, by choosing h_{t+1}, the planner is indirectly setting $d_t = \mu_t h_t e^{-1}(h_{t+1}/\mu_t h_t)$. It is easy to verify that the results are the same whether the planner chooses the variables in terms of output per unit of natural labour or per unit of efficient labour.

Turning now to the endogenous growth framework, (5) and (6) continue to hold with the same interpretation, with g_* being obtained from (9). Equation (7), however, requires a careful explanation. It points out that, along the optimal balanced growth path, the marginal benefit of an increase in the amount of output devoted to education (again per unit of efficient labour) must be equal to its marginal cost. This can be seen using the aggregate feasibility constraint (8). A rise in $\tilde{d}$ has a direct cost in terms of the third term in the RHS of (8), as it reduces consumption possibilities by $(1+n)\mu$. It also has an indirect cost, given by $\mu e'(.)(1+n)\tilde{k}$ as a consequence of the effect of a rise in $\tilde{d}$ on the rate of growth g: indeed, the greater the productivity growth rate g, the greater the amount of output that must be devoted to investment in physical capital in order to keep $\tilde{k}$ constant. However, a rise in $\tilde{d}$ (and thus in g) also has a benefit, since a greater g implies a smaller marginal rate of transformation between third and second period consumption in the LHS of (7). This amounts to an expansion of consumption possibilities that is captured by $\mu e'(.)(1+n)\tilde{c}^o_* / \left[(1+g_*)(1+n)\right]^2$. At the optimum, the marginal benefit and the marginal costs must be equal:

$$\mu(1+n)e'(\tilde{d}_*)\frac{\tilde{c}^o_*}{\left[(1+g_*)(1+n)\right]^2} = \mu(1+n)e'(\tilde{d}_*)\tilde{k}_* + \mu(1+n) \tag{10}$$

so that the terms involving $\mu(1+n)$ cancel out and (7) emerges.

Note that, in the exogenous growth framework, $\tilde{k}_*$ is univocally determined by (6) and the optimization problem can be solved sequentially. In contrast, with endogenous growth, (6) and (7) are not enough to determine $\tilde{k}_*$ and $\tilde{d}_*$ because (7) incorporates also $\tilde{c}^o_*$, which can only be obtained from (5) and (8). A sequential solution of the optimization problem is now impossible: all optimal variables are determined simultaneously.

3 Decentralized Market Equilibrium in the Presence of Government

In this section we characterize the behaviour of the economy in the presence of government. Since individuals not only decide about the allocation of their resources along their life cycle but also how much to invest in education, there are now two potential sources of inefficiencies. This is the reason why we posit that the government has two policy instruments at its disposal: education subsidies and intergenerational transfers from the middle-aged to the elderly. Among the different ways of tackling education subsidies, we choose to model them as subsidies to the repayment, in the second period of life, of the loans taken in the first one to pay for education. Let $z^m_t > 0$ [resp. < 0] be the lump-sum tax [transfer] the middle aged pay [receive], $z^o_t > 0$ [< 0] the lump-sum tax the old pay [the pension they receive] and let θ_t be the subsidy rate, all of them in period t. The laissez-faire equilibrium can be retrieved later by setting $z^m_t = z^o_t = \theta_t = 0$.[5]

[5]It is worth emphasizing that subsidies are not modelled as *direct* subsidies to expenditures in education in the first period (as in Docquier et al., 2007) but as subsidies to the repayment of education loans. Main results concerning the optimal education policy are however robust to alternative modelizations of this subsidy. See below.

Factor prices are determined under perfect competition by their marginal products, so that, if $1 + r_t$ and w_t are respectively the interest factor and the wage rate per unit of efficient labour,

$$1 + r_t = f'(\tilde{k}_t) \tag{11}$$

$$w_t = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \tag{12}$$

Individuals choose, in their first period, the amount of education that maximizes their lifetime resources. They do so by borrowing any amount they wish in perfect credit markets. Concerning savings, they behave as pure life-cyclers, i.e., they save to transfer purchasing power from the second to the third period. Then, for an individual born at $t-1$, consumption when middle-aged and consumption when old can be written respectively

$$c_t^m = w_t h_t - (1 + r_t)d_{t-1}(1 - \theta_t) - z_t^m - s_t \tag{13}$$

$$c_{t+1}^o = (1 + r_{t+1})s_t - z_{t+1}^o \tag{14}$$

where s_t are the savings of a middle-aged. Thus, the lifetime budget constraint of an individual born at period $t-1$ is:

$$c_t^m + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t h_t - (1 + r_t)d_{t-1}(1 - \theta_t) - z_t^m - \frac{z_{t+1}^o}{1 + r_{t+1}} \tag{15}$$

The first order conditions associated with the individual decision variables, d_{t-1}, c_t^m and c_{t+1}^o, are:

$$w_t e'(d_{t-1}/\overline{h}_{t-1}^y) = (1 + r_t)(1 - \theta_t) \tag{16}$$

$$\frac{\partial U(c_t^m, c_{t+1}^o)/\partial c_t^m}{\partial U(c_t^m, c_{t+1}^o)/\partial c_{t+1}^o} = (1 + r_{t+1}) \tag{17}$$

where use has been made of the homogeneity of degree 1 of the E function, i.e., $h_t = e(d_{t-1}/\overline{h}_{t-1}^y)\overline{h}_{t-1}^y$. Equation (16) shows that the individual will invest in education up to the point where the marginal benefit in terms of second period income equals the marginal cost of investing in human capital allowing for subsidies. Rewriting (16) as $e'(\tilde{d}_{t-1}) = (1 - \theta_t)\ (1 + r(\tilde{k}_t))/w(\tilde{k}_t)$, this expression implicitly characterizes the optimal ratio $\tilde{d}_{t-1}$ as a function of $\tilde{k}_t$ and θ_t, i.e., $\tilde{d}_{t-1} = \phi(\tilde{k}_t, \theta_t)$. Since $e'' < 0$ it can readily be shown that the greater $\tilde{k}_t$ and θ_t the greater $\tilde{d}_{t-1}$.

The government finances education subsidies with revenues obtained from taxing the middle-aged and/or the old-aged:

$$z_t^m L_t + z_t^o L_{t-1} = \theta_t(1 + r_t)d_{t-1}L_t \tag{18}$$

which plugged into (15) yields

$$c_t^m + \frac{c_{t+1}^o}{1 + r_{t+1}} = \omega_t \tag{19}$$

where ω_t is the present value of the net lifetime income of an individual born at $t-1$:

$$\omega_t = w_t h_t - (1 + r_t)d_{t-1}(1 - \theta_t) - z_t^m - \frac{1 + n}{1 + r_{t+1}}[\theta_{t+1}(1 + r_{t+1})d_t - z_{t+1}^m] \tag{20}$$

The homogeneity assumption on preferences implies that the c_{t+1}^o/c_t^m ratio is a function of r_{t+1} only. Substituted into the budget constraint (20) this allows us to write consumption in the second period as a fraction of lifetime income, $c_t^m = \alpha(r_{t+1})\omega_t$. Equilibrium in the market for physical capital is achieved when the (physical) capital stock available in $t+1$, K_{t+1}, equals gross savings made by the middle-aged in t, $s_t L_t$, minus the amount of output devoted to human capital investment by the young in $t, (1+n)d_t L_t$, i.e., when

$$K_{t+1} = s_t L_t - (1+n)d_t L_t \tag{21}$$

Using (21), the budget constraints (13) and (14), the government budget constraint (18) and the equilibrium factor prices (11) and (12), one can obtain the feasibility constraint (2).

In the current framework, a balanced growth path is a situation where all variables expressed in units of natural labour grow at a constant rate, with the consequence that all variables expressed in units of efficient labour remain constant over time. Therefore, $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}$, $\tilde{z}_t^m = \tilde{z}_{t+1}^m = \tilde{z}^m$, $\theta_t = \theta_{t+1} = \theta$ (and, of course, $\mu_{t+1} = \mu_t = \mu$). Factor prices will be given by (11) and (12) without time subscripts. As far as consumer behaviour is concerned, since the utility function is assumed to be homothetic, the marginal rates of substitution in the $\left(c_t^m, c_{t+1}^o\right)$ and $(\tilde{c}^m, \tilde{c}^o)$ spaces will be the same. Summing up, a balanced growth path in the presence of government intervention will fulfil the following:

$$\frac{\partial U(\tilde{c}^m, \tilde{c}^o)/\partial \tilde{c}^m}{\partial U(\tilde{c}^m, \tilde{c}^o)/\partial \tilde{c}^o} = (1+r) \tag{22}$$

$$we'(\tilde{d}) = (1+r)(1-\theta) \tag{23}$$

$$\tilde{c}^m + \frac{\tilde{c}^o}{1+r} = \hat{\omega} \tag{24}$$

where $\hat{\omega} \equiv \omega_t/h_t$ is the present value of lifetime resources expressed in terms of output per unit of efficient labour:

$$\hat{\omega} = w - \frac{(1+r)\mu\tilde{d}}{(1+g)} - \frac{[(1+r)\mu\tilde{d}\theta - (1+g)\tilde{z}^m][(1+g)(1+n) - (1+r)]}{(1+r)(1+g)} \tag{25}$$

Coming back to the equilibrium condition in the market for physical capital, and as shown in Appendix A, (21) implicitly provides $\tilde{k}_{t+1}$ as a function $\Psi(\tilde{k}_t; \tilde{z}_t^m, \tilde{z}_{t+1}^m, \theta_t, \theta_{t+1})$. Along a balanced growth path, one can delete the time subscripts and write $\tilde{k} = \Psi(\tilde{k}; \tilde{z}^m, \theta)$. An equilibrium ratio of physical capital to labour in efficiency units along a balanced growth path in the presence of government intervention, $\tilde{k}_G$, will then be a fixed point of the Ψ function, i.e., $\tilde{k}_G = \Psi(\tilde{k}_G; \tilde{z}^m, \theta)$. Such an equilibrium will be locally stable provided that $0 < \partial\Psi(\tilde{k}_G; \tilde{z}^m, \theta)/\partial\tilde{k} < 1$. In what follows, we will focus on situations where the equilibrium is unique and stable so that the relationship between $\tilde{k}$ and the tax parameters can be written, with an obvious notation, as

$$\tilde{k} = \tilde{k}(\tilde{z}^m, \theta) \tag{26}$$

We can now turn to the determination of $\tilde{d}$ or, what is the same, the growth rate g. The amount of output devoted to education per unit of inherited human capital along a balanced growth path will be governed by the relationship arising from the education decision (23), i.e., $\tilde{d} = \phi(\tilde{k}, \theta)$. Using (26) we can write $\tilde{d} = \phi\left(\tilde{k}(\tilde{z}^m, \theta), \theta\right)$ or, for short,

$$\tilde{d} = \tilde{d}(\tilde{z}^m, \theta) \tag{27}$$

Letting g_G be the growth rate of any variable expressed in terms of output per unit of natural labour, since $1+g = \mu e(\tilde{d})$, we have $1+g_G = \mu e\left(\phi(\tilde{k}_G, \theta)\right)$. Finally, the growth rate of all variables expressed in absolute terms (physical capital, human capital and output) is $(1+g_G)(1+n)$. This follows from writing H $[K]$ as hL $[kL]$ and observing that h $[k]$ grows at rate g_G while L grows at rate n.

Our ultimate concern is, however, to evaluate the effect of intergenerational transfers and education subsidies on welfare. It has already been stated that we can write individual utility as a function of consumption per unit of efficient labour, $\tilde{U} = U(\tilde{c}^m, \tilde{c}^o)$. We have also obtained the lifetime budget constraint (24). Therefore, the demands for consumption become $\tilde{c}^m = \tilde{c}^m(\hat{\omega}, r)$ and $\tilde{c}^o = \tilde{c}^o(\hat{\omega}, r)$, and the indirect utility function can be written as $V(\hat{\omega}, r)$, i.e., as a function of the present value of lifetime resources and the relative price of old age and middle-age consumption. Observe that the present value $\hat{\omega}$, in spite of being a function of a number of variables, is taken as a parameter in the indirect utility function $V(\hat{\omega}, r)$. For later use, we can obtain the partial derivatives of V with respect to its arguments $\hat{\omega}$ and r. Using (22):

$$\frac{\partial V}{\partial \hat{\omega}} = (1+r)\frac{\partial U}{\partial \tilde{c}^o}\left(\frac{\partial \tilde{c}^m}{\partial \hat{\omega}} + \frac{1}{1+r}\frac{\partial \tilde{c}^o}{\partial \hat{\omega}}\right) = (1+r)\frac{\partial U}{\partial \tilde{c}^o} \tag{28}$$

$$\frac{\partial V}{\partial r} = (1+r)\frac{\partial U}{\partial \tilde{c}^o}\left(\frac{\partial \tilde{c}^m}{\partial r} + \frac{1}{1+r}\frac{\partial \tilde{c}^o}{\partial r}\right) = \frac{\partial U}{\partial \tilde{c}^o}\frac{\tilde{c}^o}{(1+r)} \tag{29}$$

where the last equality follows in both cases from differentiation of the lifetime budget constraint (24).[6]

Coming back to function $V(\hat{\omega}, r)$, and taking into account that $\hat{\omega} = \hat{\omega}(\tilde{k}, \tilde{z}^m, \theta)$, $r = r(\tilde{k})$ and $\tilde{k} = \tilde{k}(\tilde{z}^m, \theta)$, we can write a new function that, for given values of μ and n, depends only on $\tilde{z}^m$ and θ, that is $\tilde{U} = V\left[\hat{\omega}\left(\tilde{k}(\tilde{z}^m, \theta), \tilde{z}^m, \theta\right), r\left(\tilde{k}(\tilde{z}^m, \theta)\right)\right]$. As a consequence, we end up with a new indirect utility function,

$$\tilde{U} = \tilde{V}(\tilde{z}^m, \theta) \tag{30}$$

This is the relevant function to undertake the comparative dynamics. Nonetheless, before doing that, we shall be interested in comparing the laissez-faire solution with the Golden Rule and characterizing the optimal tax parameters that support it.

4 The Laissez-Faire Balanced Growth Path and the Golden Rule

In Diamond (1965)'s model, as well as its extension to allow for exogenous productivity growth, the relationship between $\tilde{k}_*$ and $\tilde{k}_{LF}$ (LF standing for *laissez faire*) can be determined, on a one to

[6] Clearly, $\hat{\omega}$ is given at this stage. The effect of r on the present value of lifetime resources $\hat{\omega}$ will come forth when we differentiate $\hat{\omega}$ with respect to $\tilde{k}$.

	$\tilde{d}_{LF} > \tilde{d}_*$	$\tilde{d}_{LF} \leqslant \tilde{d}_*$
$\tilde{k}_{LF} \geqslant \tilde{k}_*$	$f'(\tilde{k}_{LF}) < \left(1 + g_{LF}\right)(1+n)$	not feasible
$\tilde{k}_{LF} < \tilde{k}_*$	$f'(\tilde{k}_{LF}) \lesseqgtr \left(1 + g_{LF}\right)(1+n)$	$f'(\tilde{k}_{LF}) > \left(1 + g_{LF}\right)(1+n)$

Table 1: Different laissez-faire balanced growth paths

one basis, by the relationship between $(1 + r_{LF})$ and $(1 + g_{LF})(1 + n)$. This allows to immediately evaluate whether the laissez-faire equilibrium is dynamically efficient or inefficient. As it will shortly be shown, this is no longer the case when we account for endogenous productivity growth.

This section has three main objectives. Firstly, we ask whether $\tilde{k}_{LF}$ and $\tilde{d}_{LF}$ can be respectively greater or less than $\tilde{k}_*$ and $\tilde{d}_*$ and whether over/under accumulation of physical capital can coexist with over/under accumulation of human capital. Secondly, we explore the consequences of this for the relationship between $f'(\tilde{k}_{LF})$ and $(1 + g_{LF})(1 + n)$. And, finally, it is argued that neither the relationship between $\tilde{k}_{LF}$ and $\tilde{k}_*$ nor the one between $f'(\tilde{k}_{LF})$ and $(1 + g_{LF})(1 + n)$ are enough to assess the dynamic efficiency of the laissez-faire equilibrium: $\tilde{d}_{LF}$ and $\tilde{d}_*$ need also be compared.

Since in Diamond (1965)'s model $\tilde{k}_{LF}$ may either be greater or less than $\tilde{k}_*$ (and both can, by chance, coincide), a natural conjecture is that this might be possible in the current framework as well, and, by symmetry, that $\tilde{d}_{LF}$ might also be higher/lower than (or equal to) $\tilde{d}_*$. This conjecture translates into the four possibilities associated with the four cells in Table 1 (neglecting, for the moment, the content of these cells). Note, however, that one of them is not feasible. To see this, consider a balanced growth path with $\tilde{k}_{LF} \geqslant \tilde{k}_*$. As shown in Appendix B, this situation is necessarily coupled with $\tilde{d}_{LF} > \tilde{d}_*$. Thus, using the obvious matrix notation, cell a_{11} in Table 1 is feasible, but cell a_{12} is not. Furthermore, this raises the question of whether the laissez-faire is able to support the Golden Rule, as it is indeed the case in Diamond's model for some values of the parameters. In the current framework, however, it is not the case. This follows in a straightforward way from the previous result: even if $\tilde{k}_{LF} = \tilde{k}_*$, it will be the case that $\tilde{d}_{LF} > \tilde{d}_*$, i.e., the accumulation of human capital will be too large with respect to the Golden Rule. Moreover, as it is also shown in Appendix B, $\tilde{d}_{LF} = \tilde{d}_*$ can only coexist with $\tilde{k}_{LF} < \tilde{k}_*$, i.e., with a too low accumulation of physical capital. This allows to state the following result.

Proposition 1 *The laissez-faire equilibrium cannot possibly support the Golden Rule balanced growth path.*

Coming back to our conjectures, and having discarded one of them, the other three still require verification. This can be done with the help of the Cobb-Douglas case discussed in Appendix C, where the functional forms are $f(\tilde{k}) = A\tilde{k}^{\alpha}$ $(A > 0,\ \alpha \in (0, 1))$, $e(\tilde{d}) = B\tilde{d}^{\lambda}$ $(B > 0,\ \lambda \in (0, 1))$ and $\tilde{U} = \log \tilde{c}^m + \beta \log \tilde{c}^o$ $(\beta \in (0, 1))$. The simulation results reported in Table 2 all assume $\mu = 1$, $n = 0$, $A = 1$, $B = 1$. The first three columns therein show that the combinations between laissez-faire and optimal values of $\tilde{k}$ and $\tilde{d}$ represented by cell a_{11} [case (i)], a_{21} [cases (ii)-(iv)] and a_{22}

(i)	$\alpha = .2, \beta = .9, \lambda = .1$	$\tilde{k}_{LF} > \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) < \left(1 + g_{LF}\right)(1+n)$ $e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) < 1$
(ii)	$\alpha = .2, \beta = .9, \lambda = .13$	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) < \left(1 + g_{LF}\right)(1+n)$ $e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) < 1$
(iii)	$\alpha = .25, \beta = .8, \lambda = .13$	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) > \left(1 + g_{LF}\right)(1+n)$ $e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) < 1$
(iv)	$\alpha = .3, \beta = .8, \lambda = .3$	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} > \tilde{d}_*$	$f'(\tilde{k}_{LF}) > \left(1 + g_{LF}\right)(1+n)$ $e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) > 1$
(v)	$\alpha = .1, \beta = .7, \lambda = .6$	$\tilde{k}_{LF} < \tilde{k}_*$	$\tilde{d}_{LF} < \tilde{d}_*$	$f'(\tilde{k}_{LF}) > \left(1 + g_{LF}\right)(1+n)$ $e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) > 1$

Table 2: Simulations in the Cobb-Douglas case, with $\mu = 1$, $n = 0$, $A = 1$, $B = 1$

[case (v)] in Table 1 are indeed feasible.

The next step, of course, refers to the content of the feasible cells in Table 1, i.e., the relationship between $f'(\tilde{k}_{LF})$ and $(1 + g_{LF})(1 + n)$ for each of them. To this end, we can go back to the general case and use the fact that $f'(.)$ [resp. $e(.)$] is monotonically decreasing [resp. increasing] in its argument. Thus, when $\tilde{k}_{LF} \geqslant \tilde{k}_*$ and $\tilde{d}_{LF} > \tilde{d}_*$, one has the following chain of inequalities: $f'(\tilde{k}_{LF}) \leqslant f'(\tilde{k}_*) = (1+g_*)(1+n) < (1+g_{LF})(1+n)$. From this the sign in cell a_{11} in Table 1 arises. A similar argument applies to the sign in cell a_{22}. However, this approach implies an ambiguous sign in a_{21}, since, when $\tilde{k}_{LF} < \tilde{k}_*$ and $\tilde{d}_{LF} > \tilde{d}_*$, we have $f'(\tilde{k}_{LF}) > f'(\tilde{k}_*) = (1+g_*)(1+n) < (1+g_{LF})(1+n)$. This indeterminacy can be confirmed resorting again to the Cobb-Douglas case in Table 2. As shown by cases (ii)-(iv), the relationship between the interest rate and the growth rate of the economy when $\tilde{k}_{LF} < \tilde{k}_*$ and $\tilde{d}_{LF} > \tilde{d}_*$ is ambiguous (row (ii) on the one hand and rows (iii) and (iv) on the other). This seems particularly interesting, since, in spite of having under-accumulation of physical capital with respect to the Golden Rule, the over-accumulation of human capital may give rise to a marginal product $f'(\tilde{k}_{LF})$ *lower* than the growth rate of the economy $(1 + g_{LF})(1 + n)$.

Thus far, however, we have not tried to provide a clear-cut answer to the question of whether the laissez-faire equilibrium is dynamically efficient or inefficient. An intuitive (albeit rigorous) way to address the dynamic efficiency of a balanced growth path will consist in exploring whether an expansion of the consumption possibility locus is achievable. In particular, we can evaluate the aggregate feasibility constraint (3) at the laissez-faire balanced growth path:

$$\tilde{c}^m_{LF} + \frac{\tilde{c}^o_{LF}}{\mu e(\tilde{d}_{LF})(1+n)} = f(\tilde{k}_{LF}) - (1+n)\mu\tilde{d}_{LF} - \mu e(\tilde{d}_{LF})(1+n)\tilde{k}_{LF} \tag{31}$$

and ask whether, for a given $\tilde{c}^o_{LF}$, we can increase $\tilde{c}^m_{LF}$ by changing $\tilde{k}_{LF}$ and/or $\tilde{d}_{LF}$.[7] We are then interested in evaluating $\partial\tilde{c}^m_{LF}/\partial\tilde{k}_{LF}$ and $\partial\tilde{c}^m_{LF}/\partial\tilde{d}_{LF}$. In particular, a negative sign of the first [resp. second] derivative would imply that we are able expand the feasibility constraint by reducing $\tilde{k}$ [resp. $\tilde{d}$]. If any of them is negative (or both, of course), the laissez faire equilibrium would be dynamically inefficient. Symmetrically, if both expressions are positive, the laissez-faire equilibrium will be dynamically efficient. Differentiating in (31) we obtain

$$\frac{\partial\tilde{c}^m_{LF}}{\partial\tilde{k}_{LF}} = f'(\tilde{k}_{LF}) - (1+g_{LF})(1+n) \gtreqless 0 \tag{32}$$

$$\frac{\partial\tilde{c}^m_{LF}}{\partial\tilde{d}_{LF}} = \mu(1+n)\left[e'(\tilde{d}_{LF})\left(\frac{\tilde{c}^o_{LF}}{[(1+g_{LF})(1+n)]^2} - \tilde{k}_{LF}\right) - 1\right] \gtreqless 0 \tag{33}$$

The information contained in Table 1 can be used to evaluate (32) and (33). Concerning (32), if $f'(\tilde{k}_{LF}) < (1+g_{LF})(1+n)$, a small reduction in the amount of physical capital (as measured by $\tilde{k}$) would expand consumption possibilities and welfare. With respect to (33), it is proved in Appendix B that, whenever $f'(\tilde{k}_{LF}) \leqslant (1+g_{LF})(1+n)$, a small decrease in the amount of human capital (as measured by $\tilde{d}$) would expand consumption possibilities and hence utility. Thus, if $f'(\tilde{k}_{LF}) \leqslant (1+g_{LF})(1+n)$, the laissez faire equilibrium is dynamically inefficient.

From Table 1 and the above discussion, it is now apparent that dynamic inefficiency is associated with $\tilde{d}_{LF} > \tilde{d}_*$, i.e., a too high accumulation of human capital. But, interestingly, this may coexist with either $\tilde{k}_{LF} \geqslant \tilde{k}_*$, or $\tilde{k}_{LF} < \tilde{k}_*$, i.e., with either a too high or too low accumulation of physical capital (cells a_{11} and a_{21} in Table 1). This allows to state the following proposition:

Proposition 2 *A laissez-faire equilibrium characterized by a marginal product of physical capital lower than the growth rate of the economy is dynamically inefficient, in the sense that a reduction of either physical or human capital would expand consumption possibilities and welfare. This balanced growth path is necessarily associated with an accumulation of human capital greater than that in the Golden Rule, but may be coupled with an accumulation of physical capital either greater or less than that in the Golden Rule.*

Turning now to the laissez-faire equilibria where $f'(\tilde{k}_{LF}) > (1+g_{LF})(1+n)$, it is clear from (32) that the only way to expand consumption possibilities is to *increase*, rather than decrease $\tilde{k}$. Thus, starting from the resources available along a balanced growth path, there is no scope for raising welfare by modifying the amount of physical capital. In contrast, there may be welfare improvements derived from changing the amount of *human* capital. The reason becomes apparent when one observes in Table 1 that a situation where the marginal product of physical capital exceeds the growth rate of the economy is always associated with $\tilde{k}_{LF} < \tilde{k}_*$ but may be associated with either $\tilde{d}_{LF} > \tilde{d}_*$ or $\tilde{d}_{LF} \leqslant \tilde{d}_*$ (cells a_{21} and a_{22}). In words, although the amount of physical

[7] In the model without productivity growth, or with g exogenously given, a simpler procedure suffices. Since by definition g is a constant (and $\tilde{d}$ can be forced to be zero), the LHS of (31), i.e., total consumption per unit of efficient labour, can be taken as a single entity: we can simply differentiate the whole RHS with respect to $\tilde{k}_{LF}$ to get (32).

capital will be definitely lower than that in the Golden Rule, the amount of human capital may be either higher or lower than that prevailing along the Golden Rule. As proved in Appendix B, when $f'(\tilde{k}_{LF}) > (1 + g_{LF})(1 + n)$, the derivative (33) may have any sign, with the consequence that nothing can be said with generality about the effects on welfare of a small reduction of $\tilde{d}$ on consumption possibilities and welfare along a balanced growth path. This claim is illustrated in the third, fourth and fifth rows in Table 2 for different constellations of the parameters of the production and utility functions in the Cobb-Douglas case. In all of them, the marginal product of physical capital is greater than the growth rate of the economy (and, of course, $\tilde{k}_{LF} < \tilde{k}_*$) but in cases (iii) and (iv) it is true that $\tilde{d}_{LF} > \tilde{d}_*$, whereas $\tilde{d}_{LF} < \tilde{d}_*$ in (v). More importantly, $\partial\tilde{c}^m_{LF}/\partial\tilde{d}_{LF}$ is positive in cases (iv) and (v), so that there is no scope for increasing welfare out of existing resources, but it is definitely negative in case (iii): a small reduction in $\tilde{d}$ would enhance welfare along a balanced growth path. We can thus state the following proposition

Proposition 3 *In a laissez-faire equilibrium, a marginal product of physical capital greater than the growth rate of the economy is a* necessary *but not a sufficient condition for the dynamic efficiency of the balanced growth path.*

Proposition 3 is a consequence of the fact that, in spite of the accumulation of physical capital being smaller than that in the Golden Rule, the accumulation of human capital may be either greater or less than its counterpart along the Golden Rule balanced growth path. This is equivalent to saying that, in order to assess the dynamic efficiency of the laissez-faire equilibrium in this framework, neither the relationship between $\tilde{k}_{LF}$ and $\tilde{k}_*$ nor the one between $f'(\tilde{k}_{LF})$ and $(1 + g_{LF})(1 + n)$ are enough, i.e., we also need to compare $\tilde{d}_{LF}$ and $\tilde{d}_*$. This is in contrast to what is argued by Boldrin and Montes (2005). Indeed, they claim that "a sufficient condition for the equilibrium path to be dynamically efficient is that the gross rate of return on capital be larger than or equal to one plus the growth rate of output" (p. 656), which will be the case "for reasonable values of" the parameters involved in the production and utility functions (p. 657). Proposition 3 shows that, if we adopt the Golden Rule approach, it is not a sufficient but a necessary condition. Moreover, we have found that, for (equally) reasonable values of the parameters, the laissez-faire equilibrium is dynamically inefficient in the Golden Rule sense.

5 Optimal Public Policy

We are now in a position to discuss the optimal policy, i.e., the orthopaedics that allow to convert the laissez-faire equilibrium into the optimal Golden Rule balanced growth path. Firstly, we can deal with the optimal education subsidy and then investigate the direction of the optimal intergenerational transfer (i.e., from/to the middle-aged to/from the elderly). Using (6) and (8), condition (7) can be expressed as:

$$e'(\tilde{d}_*)\left([f(\tilde{k}_*) - (1+g_*)(1+n)\tilde{k}_*] - (1+g_*)(1+n)\tilde{k}_* - (1+n)\mu\tilde{d}_* - \tilde{c}^m_*\right) = (1+g_*)(1+n) \quad (34)$$

This can be rewritten in a way that can be compared to the condition associated with the individual's choice of education in the presence of government, (23):

$$e'(\tilde{d}_*)\left(f(\tilde{k}_*) - \tilde{k}_* f'(\tilde{k}_*)\right) = f'(\tilde{k}_*)\left(1 + \frac{e'(\tilde{d}_*)\Lambda_*(\tilde{k}_*, \tilde{d}_*)}{f'(\tilde{k}_*)}\right) \tag{35}$$

with $\Lambda_*(\tilde{k}_*, \tilde{d}_*) = (1+g_*)(1+n)\tilde{k}_* + (1+n)\mu\tilde{d}_* + \tilde{c}^m_* > 0$. It is clear, from mere comparison of (23) and (35), that the optimal value of the tax parameter addressed to education, θ_*, is actually negative:

$$\theta_* = -\frac{e'(\tilde{d}_*)\Lambda_*(\tilde{k}_*, \tilde{d}_*)}{f'(\tilde{k}_*)} < 0 \tag{36}$$

This allows to state the following proposition.

Proposition 4 *Decentralizing the Golden Rule balanced growth path entails* $\theta_* < 0$*, i.e., education should be taxed.*

The intuition of this result can be easily grasped when one compares the structures of marginal benefits and marginal costs underlying the decisions of the social planner and the individual. On the one hand, the marginal benefit for the social planner from an increase in $\tilde{d}$ is the enhanced *consumption* possibilities (captured by the term involving $\tilde{c}^o_*$ in (10)) that stem from a rise in g. The marginal costs of raising $\tilde{d}$ include the direct cost (reduced consumption possibilities) and the indirect cost (requirement to increase investment in physical capital in order to keep $\tilde{k}$ constant) collected in the RHS of in (10). On the other hand, as it is clear from (23) with $\theta = 0$, in the laissez faire the marginal benefit of a unit of output invested by the individual in education in the first period is the increase in second-period *earnings*, and the marginal cost is the interest factor to be paid. When one compares (23) with $\theta = 0$ and (35) it is clear that, if the individual were confronted with the (optimal) wage and interest rates, $f(\tilde{k}_*) - \tilde{k}_* f'(\tilde{k}_*)$ and $f'(\tilde{k}_*)$, she would fail to take into account the terms collected by $\Lambda_*(\tilde{k}_*, \tilde{d}_*)$, i.e., the investment in physical capital (per unit of efficient labour) required to keep $\tilde{k}_*$ constant ($(1+g_*)(1+n)\tilde{k}_*$), its counterpart, referred to $\tilde{d}_*$ ($(1+n)\mu\tilde{k}_*$), as well as the total consumption of the middle aged (per unit of efficient labour) necessary to keep $\tilde{c}^m_*$ constant. In these circumstances, as the individual does not account for these costs, she over-invests in education and a tax is required to attain the Golden Rule.[8]

This is in sharp contrast with previous results. For example, Docquier et al. (2007) show that, when the social planner maximizes an infinite sum of utilities (with consumption measured in output per unit of natural labour), education subsidies are positive along the optimal balanced growth path. This is due to the presence of a positive intergenerational externality: when choosing their education level, individuals do not take into account that their decisions not only affect their own income but also that of their children through the inherited human capital. In our framework, when choosing how much to invest in education, the individual fails to account, not for indirect benefits of their

[8]It is worth recalling that, in our model, individuals have access to perfect credit markets where they can borrow to finance their preferred amount of education.

educational decision on future generations, but for indirect costs supported by all generations alike. Indeed, under the Golden Rule criterion, all generations obtain the same utility. These differences in results simply stem from a difference in social objectives. As we mentioned in the introduction, the Golden Rule approach is most pertinent if we think that, as productivity increases, consumption per unit of natural labour (and, consequently, both physical and human capital per unit of natural labour) has to increase at the same rate in order to maintain well-being. If we adopt this view, education should be *taxed* along the optimal balanced growth path. Nevertheless, as discussed in the next section, positive education subsidies can be welfare improving under certain conditions out of the Golden Rule balanced growth path.

It is worth mentioning that Proposition 4 does not depend on the specific way education subsidies are modelled, i.e., subsidies to the repayment of the loans. As we have seen, this result is directly driven from the optimality conditions and makes no use of the government budget constraint. Thus, as noted previously (see footnote 5) the fact that education should be taxed is independent of the way we model government intervention in education investments.[9]

We can now turn to the issue of the direction of the optimal intergenerational transfers, and, in particular, whether they adopt the form of positive or negative lump-sum taxes on the older generation. As shown in Appendix D, a clear-cut result can be obtained, which can be summarized in the following proposition.

Proposition 5 *Decentralizing the Golden Rule balanced growth path involves* $\tilde{z}_*^o < 0$*, i.e., positive pensions to the elderly.*

It should be stressed that, in contrast to the result referred to the optimal education tax, the sign of the optimal transfers to the old-aged underlying Proposition 5 is not robust, and depends crucially on the way public finances are modelled.

6 Comparative Dynamics

The preceding section has characterized the sign of the tax parameters that allow to decentralize the Golden Rule balanced growth path. The current one emphasizes a related but conceptually different issue: the comparative dynamics, in terms of (both physical and human) capital accumulation and welfare, of variations in the level of intergenerational transfers and the tax parameter addressed to education decisions, when the economy follows an arbitrary balanced growth path.

The relevant functional forms to address the comparative dynamics are (26), (27) and (30). Starting with the first of them, as argued in Section 3 and shown in Appendix A, it is implicitly

[9]This means that Proposition 4 holds when the government directly subsidizes education expenditures in the first period, as in Docquier et al. (2007). The counterpart of the government budget constraint (18) is now: $z_t^m L_t + z_t^o L_{t-1} = \sigma_t d_t L_{t+1}$ where σ_t is the subsidy rate to the education expenditures of the L_{t+1} members of the younger generation. This has obvious consequences for the individual budget constraint and the equilibrium condition in the market for physical capital, but does not affect the result that, along the Golden Rule balanced growth path, the optimal education subsidy is negative, i.e., it is a tax.

given by $\tilde{k} = \Psi(\tilde{k}; \tilde{z}^m, \theta)$, so that:

$$\frac{\partial \tilde{k}(\tilde{z}^m, \theta)}{\partial \tilde{z}^m} = \frac{\partial \Psi(.)}{\partial \tilde{k}} \frac{\partial \tilde{k}(\tilde{z}^m, \theta)}{\partial \tilde{z}^m} + \frac{\partial \Psi(.)}{\partial \tilde{z}^m} = \frac{1}{\Omega} \left[\frac{-(1-\alpha(r))}{(1+n)\mu e(\phi(.))} - \frac{\alpha(r)}{(1+r)} \right] < 0 \tag{37}$$

where $0 < \Omega = 1 - \partial \Psi(.)/\partial \tilde{k} < 1$ along a locally stable balanced growth path. In words, (37) implies that the larger are the taxes paid by the middle-aged, other things being the same, the lower is the $\tilde{k}$ ratio. Thus, the message emerging from exogenous growth models with a pure life-cycle saving motive continues to hold in the current framework: intergenerational transfers from the middle aged to the elderly depress savings and physical capital accumulation (as measured by $\tilde{k}$). Now, we can turn to expression $\tilde{d} = \phi\left(\tilde{k}(\tilde{z}^m, \theta), \theta\right)$, which, as discussed in Section 3, implicitly characterizes (27). Since $\partial \phi / \partial \tilde{k} > 0$, it follows that $\partial \tilde{d}(\tilde{z}^m, \theta)/\partial \tilde{z}^m = (\partial \phi(.)/\partial \tilde{k})(\partial \tilde{k}(.)/\partial \tilde{z}^m) < 0$. Therefore, the above result concerning physical capital can be extended to human capital: transfers from younger to older generations not only reduce the accumulation of physical capital (as measured by $\tilde{k}$) but also the accumulation of human capital (as measured by $\tilde{d}$) and the growth rate of the economy.

As far as the effects of changes in the education subsidy are concerned, one would be tempted to advance that an increased θ will translate into a higher value of $\tilde{d}$. As $\partial \phi / \partial \theta > 0$, this is certainly the case when $\tilde{k}$ is held constant in the expression characterizing the individual's decision on education, $\tilde{d} = \phi(\tilde{k}, \theta)$. However, this is nothing else but a partial-equilibrium result that neglects the effects of the tax subsidy on $\tilde{k}$. As it will be seen shortly, when these effects are taken into account, it is impossible to say in general whether higher education subsidies will have a positive or a negative effect on the accumulation of human capital (as measured by $\tilde{d}$) and the economy's growth rate. The reason for this can be found in that

$$\begin{aligned} \frac{\partial \tilde{k}(\tilde{z}^m, \theta)}{\partial \theta} &= \frac{\partial \Psi(.)}{\partial \tilde{k}} \frac{\partial \tilde{k}(\tilde{z}^m, \theta)}{\partial \theta} + \frac{\partial \Psi(.)}{\partial \theta} = \frac{1}{\Omega} \left[\frac{(1-\alpha)(1+r)\phi(.)}{\mu e[\phi(.)]^2 (1+n)} + \frac{\alpha \phi(.)}{e[\phi(.)]} \right] \\ & -\frac{1}{\Omega} \left[\frac{(1-\alpha)e'(.)}{\mu e[\phi(.)]^2 (1+n)} \frac{\partial \phi(.)}{\partial \theta} \left(w - \frac{(1+r)(1-\theta)\phi(.)}{e[\phi(.)]} - \tilde{z}^m \right) \right] \\ & -\frac{1}{\Omega} \left[\left(\frac{(1-\alpha)(1+r)(1-\theta)}{\mu e[\phi(.)]^2 (1+n)} + \frac{(1-\alpha\theta)}{e[\phi(.)]} \right) \frac{\partial \phi(.)}{\partial \theta} \left(1 - \frac{e'(.)}{e(.)/\phi(.)} \right) \right] \end{aligned} \tag{38}$$

where the first two terms in the second part of (38) are positive while the two latter are negative. This indeterminacy reflects the complexity of the underlying interaction between the subsidy rate and the consumption and education decisions, and implies that the effect of the subsidy on the $\tilde{d}$ ratio is also ambiguous:

$$\frac{\partial \tilde{d}(\tilde{z}^m, \theta)}{\partial \theta} = \frac{\partial \phi(.)}{\partial \tilde{k}} \frac{\partial \tilde{k}(\tilde{z}^m, \theta)}{\partial \theta} + \frac{\partial \phi(.)}{\partial \theta} \tag{39}$$

The previous results can be summarized in the following proposition.

Proposition 6 *When the starting point is any arbitrary balanced growth path, a lump-sum transfer from the middle-aged to the elderly translates into a smaller accumulation of both physical and human capital (and thus a smaller growth rate), but a change in the rate of education subsidy may have either a positive or negative effect on both.*

We can now focus on the welfare effects, along a balanced growth path, of changing the tax parameters. The conclusion obtained in overlapping generations models with exogenous growth is well known: when the marginal product of physical capital is lower [resp. higher] than the economy's growth rate (reflecting over [resp. under] accumulation of physical capital), a lump-sum transfer from [resp. to] younger to [resp. from] older generations provides a means to bring the economy closer to the Golden Rule. However, as our analysis will show, in the current model where human capital is the engine of growth, this condition is not enough any longer, and further requirements are to be met. Using the indirect utility function $\tilde{U} = V\left[\hat{\omega}\left(\tilde{k}(\tilde{z}^m,\theta),\tilde{z}^m,\theta\right), r\left(\tilde{k}(\tilde{z}^m,\theta)\right)\right]$, we have:

$$\frac{\partial \tilde{V}(\tilde{z}^m,\theta)}{\partial \tilde{z}^m} = \frac{\partial V}{\partial \hat{\omega}}\left(\frac{\partial \hat{\omega}}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial \tilde{z}^m} + \frac{\partial \hat{\omega}}{\partial \tilde{z}^m}\right) + \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\tilde{k}}\frac{\partial \tilde{k}}{\partial \tilde{z}^m} \tag{40}$$

with a similar expression for the changes in θ. As shown in Appendix E, using (28), (29), the lifetime budget constraint and the equilibrium condition in the market for physical capital, (40) can be rewritten:

$$\frac{\partial \tilde{V}(\tilde{z}^m,\theta)}{\partial \tilde{z}^m} = \frac{\partial U}{\partial \tilde{c}^o}\left(J\left[(1+g)(1+n)-(1+r)\right] + M\frac{\partial \hat{\omega}}{\partial \tilde{d}}\right)\frac{\partial \tilde{k}}{\partial \tilde{z}^m} + \frac{\partial U}{\partial \tilde{c}^o}\left[(1+g)(1+n)-(1+r)\right] \tag{41}$$

where $J = \tilde{k}f'' + \mu\phi(.)(1-\theta)f''/(1+g)$ is negative, $M = (1+r)\mu(\partial\phi(.)/\partial\tilde{k})$ is positive and

$$\frac{\partial \hat{\omega}}{\partial \tilde{d}} = -\frac{(1+r)}{(1+g)} + \frac{(1+r)\mu\phi(.)(1-\theta)e'(.)}{(1+g)^2} + \frac{\tilde{z}^m(1+n)e'(.)}{1+r} - \frac{\theta\left[(1+g)(1+n)-(1+r)\right]}{(1+g)} \tag{42}$$

In order to interpret (41), it is important to realize that $\partial\hat{\omega}/\partial\tilde{d}$ captures the effect of a change in the accumulation of human capital (as measured by $\tilde{d}$) on the present value of the individual's lifetime resources for given values of w, r, $\tilde{z}^m$ and θ. This can easily be checked by partial differentiation of (25). It can also be verified that this expression cannot be signed in general. Overall, the first term in the RHS of (41) reflects the effect of the change in $\tilde{k}$ induced by the lump-sum tax on the middle aged, while the second term reflects the direct effect of $\tilde{z}^m$ on welfare. Clearly, if $[(1+g)(1+n)-(1+r)]$ and $\partial\hat{\omega}/\partial\tilde{d}$ have opposite signs, (41) can be signed without ambiguity, and we can enunciate the following proposition.

Proposition 7 *A lump-sum transfer from the middle-aged to the elderly entails a welfare increase [resp. decrease] along a balanced growth path provided that: (i) the economy's growth rate is greater [resp. less] than the interest rate, and (ii) investing in education reduces [resp. increases], at the margin, the present value of the individual's lifetime resources. Otherwise, the effects of intergenerational transfers on welfare along the balanced growth path are ambiguous.*

The intuition is simple and highlights the importance of both (i) and (ii) being simultaneously fulfilled. As for (i), it is actually what arises from an exogenous growth model when $\tilde{z}^m$ and $\tilde{z}^o$ are respectively interpreted as the tax paid and the pension received in a pure pay-as-you-go social security system: if $(1+g)(1+n) > (1+r)$, so that the rate of return of "investing in future generations" exceeds that of investing in physical capital, an additional amount paid to social security will expand consumption possibilities and welfare. But, in an endogenous growth setting, the above condition must also be accompanied by (ii): since increasing $\tilde{z}^m$ depresses $\tilde{k}$ and thus discourages investments in $\tilde{d}$, welfare will unambiguously increase along the balanced growth path only if, in addition to $(1+g)(1+n) > (1+r)$, the reduced human capital accumulation translates into a greater lifetime income, i.e., $\partial\hat{\omega}/\partial\tilde{d} < 0$. The argument should be accordingly reversed when the starting point entails $(1+g)(1+n) < (1+r)$ and $\partial\hat{\omega}/\partial\tilde{d} > 0$. It is important to stress that, if $[(1+g)(1+n)-(1+r)]$ and $\partial\hat{\omega}/\partial\tilde{d}$ have the same sign, the final effect is indeterminate.

Following the same steps, the effects of an increase in the education subsidy rate on welfare along a balanced growth path can be summarized in the following expression:

$$\begin{aligned}\frac{\partial\tilde{V}(\tilde{z}^m,\theta)}{\partial\theta} &= \frac{\partial U}{\partial\tilde{c}^o}\left(J\left[(1+g)(1+n)-(1+r)\right]+M\frac{\partial\hat{\omega}}{\partial\tilde{d}}\right)\frac{\partial\tilde{k}}{\partial\theta} \\ &\quad -\frac{\partial U}{\partial\tilde{c}^o}N\left[(1+g)(1+n)-(1+r)\right]+\frac{\partial U}{\partial\tilde{c}^o}P\frac{\partial\hat{\omega}}{\partial\tilde{d}} \end{aligned} \tag{43}$$

where $N = (1+r)\mu\phi(.)/(1+g)$ and $P = (1+r)\mu(\partial\phi(.)/\partial\theta)$ are both positive. It is clear from mere inspection that the first terms in the RHS of (41) and (43) have the same structure, i.e., they capture the effect on $\tilde{k}$ induced by a small change in a tax parameter. But this should not obscure the fact that while $\partial\tilde{k}/\partial\tilde{z}^m$ can be signed, $\partial\tilde{k}/\partial\theta$ cannot. Indeed, only if (38) happens to be positive, (43) will also have a clear-cut sign whenever $[(1+g)(1+n)-(1+r)]$ and $\partial\hat{\omega}/\partial\tilde{d}$ have opposite signs, i.e.,

Proposition 8 *Provided that education subsidies foster physical capital accumulation, they will also increase [resp. decrease] welfare along a balanced growth path whenever: (i) the economy's growth rate is lower [resp. higher] than the interest rate, and (ii) investing in education increases [resp. decreases], at the margin, the present value of the individual's lifetime resources.*

The intuition underlying this result is similar to the one in Proposition 7. But in addition to the requirement that $[(1+g)(1+n)-(1+r)]$ and $\partial\hat{\omega}/\partial\tilde{d}$ have opposite signs, the exigency that $\partial\tilde{k}/\partial\theta > 0$ now becomes crucial. If, instead, $\partial\tilde{k}/\partial\theta < 0$, the sign of (43) is ambiguous and nothing can be said with generality.

To conclude, we can relate the discussion of the comparative dynamics to the one in Section 4 on optimal tax policy. Optimal intergenerational transfers, $\tilde{z}^m_*$, and education subsidies/taxes, θ_*, will be simultaneously determined by $\partial\tilde{V}(.)/\partial\tilde{z}^m = 0$ and $\partial\tilde{V}(.)/\partial\theta = 0$. Using (41) and (43), it is clear that $(1+g)(1+n) = (1+r)$ and $\partial\hat{\omega}/\partial\tilde{d} = 0$ provide a solution to this system of equations. The first of them is nothing else but (6), i.e., the equality between the marginal product of physical capital, $f'(\tilde{k}_*)$, and the growth rate of the economy, $(1+g_*)(1+n)$, along the Golden Rule balanced growth path. The second one does not seem to be so obvious. However, after some manipulation, using the individual budget constraint and the physical capital market equilibrium condition, the expression

$\partial\hat{\omega}/\partial\tilde{d} = 0$ transforms into (7) when evaluated at the Golden Rule. In sum, we are back to the discussion and the results in Section 4: along the Golden Rule balanced growth path, education should be taxed and the members of the older generation should receive a positive pension.

7 Concluding comments

This paper has focused on the relationship among education subsidies, a scheme of intergenerational transfers and welfare in a life-cycle growth model with both physical and human capital. The welfare objective has been taken to be the Golden Rule, i.e., the balanced growth path that maximizes the lifetime welfare of a representative individual subject to the feasibility constraint and the requirement that everyone else's welfare is fixed at the same level. And we have done so in the only sensible way in an endogenous growth framework, i.e., by considering a utility function whose arguments are individual consumptions measured in terms of output per unit of efficient (instead of natural) labour. We have compared the physical and human capital to efficient labour ratios under the laissez faire with perfect credit markets and the Golden Rule, and it has been shown that whether the rate of return to physical capital is higher or lower than the growth rate of the economy is not enough to assess the efficiency of the competitive equilibrium. The reason can be found in the fact that, even if physical capital investments are too low at the laissez-faire economy, human capital investments can be too large.

We have also shown that the laissez-faire equilibrium cannot possibly support the Golden Rule balanced growth path. Actually, along the Golden Rule balanced growth path, the optimal education subsidy is negative, i.e., the social planner should be *taxing* education investments. The intuition is that, if individuals faced the optimal (i.e., Golden Rule) wage and interest rates in a laissez faire framework, they would dismiss some of the ensuing costs and would over-invest in education. Nevertheless, under certain conditions, positive education subsidies may still be welfare improving when the starting point is a non-optimal arbitrary balanced growth path. As far as intergenerational transfers are concerned, when the tax parameter addressed to education is related to loan repayments, a strong case can be made for the existence of a positive transfer to old-aged individuals (i.e., *positive* pensions).

To conclude, it is clear that some of the assumptions underlying the model, particularly that of perfect credit markets, are not realistic and must somehow be relaxed. Indeed, policy conclusions may be quite sensitive not only to whether or not individuals face constraints when trying to borrow to finance their education investments but also to the reasons for these constraints. It seems fair to say that more research on this subject is warranted.

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Appendix

A. Derivation of equation $\tilde{k}_{t+1} = \Psi(\tilde{k}_t; \tilde{z}^m_t, \tilde{z}^m_{t+1}, \theta_t, \theta_{t+1})$

The equilibrium condition in the market for physical capital, given by (21), can be written

$$K_{t+1} = \left\{(1-\alpha(r_{t+1}))\,\omega_t + (1+n)\,\theta_{t+1}d_t - \frac{(1+n)\,z^m_{t+1}}{1+r_{t+1}}\right\} L_t - (1+n)d_tL_t \tag{A.1}$$

and dividing by L_t it can be expressed in units of natural labour:

$$k_{t+1} = \frac{(1-\alpha(r_{t+1}))\,\omega_t}{1+n} - \frac{z^m_{t+1}}{1+r_{t+1}} - d_t(1-\theta_{t+1}) \tag{A.2}$$

with ω_t given by (20). Writing $d_t = \phi(\tilde{k}_{t+1}, \theta_{t+1})\overline{h}^y_t$, dividing by h_t and taking into account that $h_{t+1} = e\left(\phi(\tilde{k}_{t+1}, \theta_{t+1})\right)\mu_t h_t$, we can manipulate and rewrite (A.2) as:

$$\tilde{k}_{t+1} = \frac{(1-\alpha(r_{t+1}))\,\hat{\omega}_t}{\mu_t e(\phi(\tilde{k}_{t+1}, \theta_{t+1}))(1+n)} - \frac{\tilde{z}^m_{t+1}}{1+r_{t+1}} - \frac{(1-\theta_{t+1})\,\phi(\tilde{k}_{t+1}, \theta_{t+1})}{e(\phi(\tilde{k}_{t+1}, \theta_{t+1}))} \tag{A.4}$$

This expression implicitly provides $\tilde{k}_{t+1}$ as a function of $\tilde{k}_t$, $\tilde{z}^m_t, \tilde{z}^m_{t+1}, \theta_t$, θ_{t+1}.

B. Formal proofs corresponding to Section 4

We first study the different possibilities regarding, on the one hand, the relationship between $\tilde{k}_{LF}$ and $\tilde{k}_*$ and, on the other, $\tilde{d}_{LF}$ and $\tilde{d}_*$. Also, we address whether the laissez-faire equilibrium can be optimal. Using (6) and (8) we can rewrite (7) as

$$e'(\tilde{d}_*)\left(f(\tilde{k}_*) - \tilde{k}_* f'(\tilde{k}_*) - \Lambda_*(\tilde{k}_*, \tilde{d}_*)\right) = f'(\tilde{k}_*) \tag{B.1}$$

with $\Lambda_*(\tilde{k}_*, \tilde{d}_*) = (1+g_*)\,(1+n)\tilde{k}_* + (1+n)\mu\tilde{d}_* + \tilde{c}^m_* > 0$. This optimality condition can now be readily compared to the one characterizing the laissez-faire choice of education (23) with $\theta = 0$ and using factor prices in (11) and (12):

$$e'(\tilde{d}_{LF})\left(f(\tilde{k}_{LF}) - \tilde{k}_{LF} f'(\tilde{k}_{LF})\right) = f'(\tilde{k}_{LF}) \tag{B.2}$$

Suppose that $\tilde{k}_{LF} \geqslant \tilde{k}_*$, then $f'(\tilde{k}_{LF}) \leqslant f'(\tilde{k}_*)$ and $f(\tilde{k}_{LF}) - \tilde{k}_{LF}f'(\tilde{k}_{LF}) \geqslant f(\tilde{k}_*) - \tilde{k}_*f'(\tilde{k}_*)$. One then obtains, letting $\Sigma(\tilde{k}) = f'(\tilde{k})/(f(\tilde{k}) - \tilde{k}f'(\tilde{k}))$,

$$e'(\tilde{d}_{LF}) = \Sigma(\tilde{k}_{LF}) \leqslant \Sigma(\tilde{k}_*) < \Sigma(\tilde{k}_*) + \Lambda_* e'(\tilde{d}_*) = e'(\tilde{d}_*) \tag{B.3}$$

and, therefore, $\tilde{d}_{LF} > \tilde{d}_*$. Hence, $\tilde{k}_{LF} \geqslant \tilde{k}_*$ is not compatible with $\tilde{d}_{LF} \leqslant \tilde{d}_*$, implying that cell a_{12} in Table 1 is not feasible. It also follows that, even if it happens that $\tilde{k}_{LF} = \tilde{k}_*$, it will be the case that $\tilde{d}_{LF} > \tilde{d}_*$. Consequently, even if the accumulation of physical capital is optimal, the accumulation of human capital will not. To show that the laissez faire will never be able to reach the optimal allocation, it suffices to show that, even if the accumulation of human capital is optimal, the accumulation of physical capital will not. Indeed, suppose that $\tilde{d}_{LF} = \tilde{d}_*$, then

$$\Sigma(\tilde{k}_{LF}) = e'(\tilde{d}_{LF}) = e'(\tilde{d}_*) = \Sigma(\tilde{k}_*) + \Lambda_* e'(\tilde{d}_*) > \Sigma(\tilde{k}_*) \tag{B.4}$$

and this necessarily implies that $\tilde{k}_{LF} < \tilde{k}_*$.

If we now consider the case where $\tilde{k}_{LF} < \tilde{k}_*$, so that $\Sigma(\tilde{k}_{LF}) > \Sigma(\tilde{k}_*)$, following the same steps as before, we get

$$e'(\tilde{d}_{LF}) = \Sigma(\tilde{k}_{LF}) \lesseqgtr \Sigma(\tilde{k}_*) + \Lambda_* e'(\tilde{d}_*) = e'(\tilde{d}_*) \tag{B.5}$$

from which $\tilde{d}_{LF} \lesseqgtr \tilde{d}_*$. This indeterminacy is illustrated in Table 2 by simulations (ii)-(iv) on the one hand and (v) on the other.

Secondly, in order to address the dynamic efficiency or inefficiency of the laissez-faire equilibrium, we can evaluate the sign of (33). Using (8), one has

$$e'(\tilde{d}_{LF}) \left(f(\tilde{k}_{LF}) - (1 + g_{LF})(1 + n)\tilde{k}_{LF} - \Lambda_{LF}(\tilde{k}_{LF}, \tilde{d}_{LF}) \right) \gtreqless (1 + g_{LF})(1 + n) \tag{B.6}$$

where $\Lambda_{LF}(\tilde{k}_{LF}, \tilde{d}_{LF}) = (1 + g_{LF})(1+n)\tilde{k}_{LF} + (1+n)\mu\tilde{d}_{LF} + \tilde{c}_{LF} > 0$. In order to account for the fact that in general at the laissez-faire $f'(\tilde{k}_{LF}) \gtreqless (1 + g_{LF})(1+n)$ we write $(1 + g_{LF})(1+n) = f'(\tilde{k}_{LF}) + \varepsilon$ for $\varepsilon \gtreqless 0$. Then

$$e'(\tilde{d}_{LF}) \left(f(\tilde{k}_{LF}) - \left(f'(\tilde{k}_{LF}) + \varepsilon \right) \tilde{k}_{LF} - \Lambda_{LF}(\tilde{k}_{LF}, \tilde{d}_{LF}) \right) \gtreqless \left(f'(\tilde{k}_{LF}) + \varepsilon \right) \tag{B.7}$$

that simplifies to

$$e'(\tilde{d}_{LF}) \left(-\varepsilon\tilde{k}_{LF} - \Lambda_{LF}(\tilde{k}_{LF}, \tilde{d}_{LF}) \right) \gtreqless \varepsilon \tag{B.8}$$

Therefore, if $\varepsilon \geqslant 0$ (i.e., $f'(\tilde{k}_{LF}) \leqslant (1 + g_{LF})(1 + n)$) the sign of (B.8) is negative and we can conclude that, whenever $f'(\tilde{k}_{LF}) \leqslant (1 + g_{LF})(1 + n)$ the equilibrium is dynamically inefficient. Note, in Table 1, that this result applies even if $\tilde{k}_{LF} < \tilde{k}_*$, as long as $f'(\tilde{k}_{LF}) \leqslant (1 + g_{LF})(1 + n)$. However, if $\varepsilon > 0$ (i.e., $f'(\tilde{k}_{LF}) > (1 + g_{LF})(1 + n)$) the sign of (B.8) is ambiguous in general. Although according to the table, this situation can only correspond to $\tilde{k}_{LF} < \tilde{k}_*$, there is still scope for a Pareto improvement provided that the sign of (B.8) is negative. This is possible, as illustrated by simulation (iii) in Table 2.

C. The Cobb-Douglas Case

Consider the functional forms presented in section 4, $f(\tilde{k}) = A\tilde{k}^{\alpha}$, $e(\tilde{d}) = B\tilde{d}^{\lambda}$ and $\tilde{U} = \log \tilde{c}^m + \beta \log \tilde{c}^o$. The optimal distribution of consumption (5) writes

$$\frac{\tilde{c}^o_*}{\beta\tilde{c}^m_*} = (1 + g_*)(1 + n) \tag{C.1}$$

and (9) is $1 + g_* = \mu B\tilde{d}^{\lambda}_*$. Using this, and (C.1) we can write (8)

$$\tilde{c}^o_* = \frac{\beta}{1+\beta} \left(A\tilde{k}^{\alpha}_* - (1+n)\mu B\tilde{d}^{\lambda}_*\tilde{k}_* - (1+n)\mu\tilde{d}_* \right) (1+n)\mu B\tilde{d}^{\lambda}_* \tag{C.2}$$

Also, $f'(\tilde{k}_*) = \alpha A \tilde{k}_*^{\alpha-1}$ and $e'(\tilde{d}_*) = \lambda B \tilde{d}_*^{\lambda-1}$, so that (7) becomes

$$\begin{aligned}&\lambda B\tilde{d}_*^{\lambda-1}\left[\beta\left(A\tilde{k}_*^{\alpha}-(1+n)\mu B\tilde{d}_*^{\lambda}\tilde{k}_*-(1+n)\mu\tilde{d}_*\right)-(1+\beta)(1+n)\mu B\tilde{d}_*^{\lambda}\tilde{k}_*\right]=\\&=(1+\beta)(1+n)\mu B\tilde{d}_*^{\lambda}\end{aligned} \tag{C.3}$$

while (6) is $\alpha A \tilde{k}_*^{\alpha-1} = (1+n)\mu B \tilde{d}_*^{\lambda}$. These two equations, allow to characterize $(\tilde{k}_*, \tilde{d}_*)$.

With respect to the laissez-faire, note first that savings are $s_t = \frac{\beta}{1+\beta}\omega_t$ and, from (23), the optimal choice of education is $d_{t-1} = \frac{\lambda(1-\alpha)}{\alpha}k_t$. This allows to write the physical capital market equilibrium condition (21) as a non-linear first-order differential equation in $\tilde{k}$, i.e., $\tilde{k}_{t+1} = G(.)\tilde{k}_t^{\alpha(1-\lambda)}$, where $G(.)$ is a constant that depends on α, β, λ, n, μ, A and B. The balanced growth path with a positive value of $\tilde{k}$ is unique and globally stable and is given by

$$\tilde{k}_{LF} = \left[\frac{\left(\frac{1}{(1+n)}\frac{\beta}{(1+\beta)}\frac{\alpha(1-\alpha)(1-\lambda)A}{\alpha+\lambda(1-\alpha)}\right)^{1-\lambda}}{\left(\frac{\lambda(1-\alpha)}{\alpha}\right)^{\lambda}\mu^{1-\lambda}B}\right]^{\frac{1}{1-\alpha(1-\lambda)}} \tag{C.4}$$

The associated value of $\tilde{d}$ can be found from the optimal education choice and writes

$$\tilde{d}_{LF} = \left[\frac{B\lambda(1-\alpha)}{\alpha}\right]^{\frac{1}{1-\lambda}}\left(\tilde{k}_{LF}\right)^{\frac{1}{1-\lambda}} \tag{C.5}$$

Using (C.4) and (C.5) one can construct the simulation results reported in Table 2.

D. Proof of Proposition 5

The equilibrium condition in the market for physical capital, (21), can be rewritten along a balanced growth path as:

$$(1+g)(1+n)\tilde{k} = w - \frac{1+r}{1+g}\mu\tilde{d}(1-\theta) - \tilde{z}^m - \tilde{c}^m - (1+n)\mu\tilde{d} \tag{D.1}$$

On the other hand, from the individual budget constraint (24), and using (25) and (D.1):

$$\frac{\tilde{c}^o}{(1+r)^2} = \frac{1}{1+r}\left((1+g)(1+n)\tilde{k} + (1+n)(1-\theta)\mu\phi(.) + \tilde{z}^m\frac{(1+g)(1+n)}{(1+r)}\right) \tag{D.2}$$

We now turn to (7). Evaluating (D.2) at the Golden Rule, i.e. for the optimal tax parameters $\tilde{z}_*^m$ and θ_*, and making use of the fact that $1+g_* = \mu e(\tilde{d}_*)$, we obtain:

$$\frac{e'(\tilde{d}_*)}{e(\tilde{d}_*)/\tilde{d}_*} - 1 = e'(\tilde{d}_*)\left(\frac{\mu\tilde{d}_*\theta_*}{1+g_*} - \frac{\tilde{z}_*^m}{(1+g_*)(1+n)}\right) < 0 \tag{D.3}$$

By the concavity of $e(.)$, $e'(\tilde{d}_*) < e(\tilde{d}_*)/\tilde{d}_*$ so that the LHS of this expression is negative. It then follows that the expression in parenthesis on the RHS is also negative. Now, writing the government budget constraint (18) in units of efficienct labour yields $\tilde{z}^m + \tilde{z}^o/[(1+g)(1+n)] = \theta(1+r)\mu\tilde{d}/(1+g)$. Then, at the Golden Rule:

$$\frac{\tilde{z}^o_*}{[(1+g_*)(1+n)]^2} = \frac{\mu\tilde{d}_*\theta_*}{1+g_*} - \frac{\tilde{z}^m_*}{(1+g_*)(1+n)} \tag{D.4}$$

which is negative by (D.3), thus implying that $\tilde{z}^o_*$ must be negative, i.e., pensions must be positive.

E. Welfare effects of modifying the tax parameters

The comparative dynamics in terms of welfare along a balanced growth path are given by $\partial\tilde{V}(.)/\partial\tilde{z}^m$ and $\partial\tilde{V}(.)/\partial\theta$. Starting from (40), note first that $\partial V/\partial\hat{\omega}$ and $\partial V/\partial r$ are respectively provided by (28) and (29). In (29) we substitute $\tilde{c}^o/(1+r)$ from (D.2). Second, $\partial\tilde{k}/\partial\tilde{z}^m$ is given by (37) and $\mathrm{d}r/\mathrm{d}\tilde{k} = f''(\tilde{k})$. Third, from (25)

$$\begin{aligned}\frac{\partial\hat{\omega}}{\partial\tilde{k}} = &-\frac{\left(\mu\phi(.)f''(.) + (1+r)\mu\frac{\partial\phi(.)}{\partial\tilde{k}}\right)(1-\theta)(1+g) - (1+r)(1-\theta)\mu\phi(.)\mu e'(.)\frac{\partial\phi(.)}{\partial\tilde{k}}}{(1+g)^2} - \\ &- \tilde{z}^m\frac{\left(f''(.) - \mu e'(.)\frac{\partial\phi(.)}{\partial\tilde{k}}(1+n)\right)(1+r) - f''(.)\left[(1+r)-(1+g)(1+n)\right]}{(1+r)^2} - \\ &- \tilde{k}f''(.) - (1+n)\theta\mu\frac{\partial\phi(.)}{\partial\tilde{k}}\end{aligned} \tag{E.1}$$

and

$$\frac{\partial\hat{\omega}}{\partial\tilde{z}^m} = -\frac{[(1+r)-(1+g)(1+n)]}{1+r} \tag{E.2}$$

Rearranging terms, we obtain (41).

Similarly, in the counterpart of (40) referred to θ, $\partial\tilde{k}/\partial\theta$ is given by (38) and, from (25),

$$\begin{aligned}\frac{\partial\hat{\omega}}{\partial\theta} = &\frac{\left(\mu\phi(.) + \mu\frac{\partial\phi(.)}{\partial\theta}\theta\right)[(1+r)-(1+g)(1+n)]}{(1+g)} + \\ &+ \mu\frac{\partial\phi(.)}{\partial\theta}\left[-\frac{(1+r)}{(1+g)} + \frac{(1+r)\mu\phi(.)e'(.)(1-\theta)}{(1+g)^2} + \frac{\tilde{z}^m(1+n)e'(.)}{1+r}\right]\end{aligned} \tag{E.3}$$

Rearranging terms, (43) results.